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EFFECTS OF STELLAR ENCOUNTERS ON PLANETARY MOTIONS ~~NASA~~ CR 56232

by

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Summary

Changes in the semi-major axis, eccentricity and inclination of a planetary orbit by an encounter with a passing star are calculated by the method of variation of elements. In order to estimate the cumulative effects of a long series of encounters, the velocity distribution of the stars is assumed to be Gaussian, and use is made of the central-limit theorem of probability. Explicit forms for the standard deviations of the distributions of the changes of the elements are obtained in terms of the orbital elements of the planetary orbits, the average stellar velocity  $v_m$ , the mean stellar density  $\nu$ , the age of the system  $T$ , and the lower bound of the impact parameter for which the analytical methods apply. If  $v_m = 20 \text{ km sec}^{-1}$ ,  $\nu = 0.1 \text{ star parsec}^{-3}$ , and  $T = 4 \times 10^9 \text{ years}$ , and passing stars with impact parameter greater than some ten times the semi-major axis of the planet are taken into account, the cumulative effects are found to be small. For very close encounters, direct numerical integrations of the equations of motion are carried out and some examples of actual expulsion of a planet are found. These results provide further confirmation of the modern result that both capture and disruption can occur in the problem of three bodies.

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1. Introduction. - During the whole past history of the solar system several influences will have been operative to affect the sizes, shapes, and mutual inclinations of the planetary orbit. On the purely dynamical non-dissipative side there will be the interactions between planets themselves, and with the asteroids and comets, and with passing stars, while on the dissipative side there will be the effect of inter-planetary gas and dust, meteoritic impacts with planets and satellites, and the effects of passage of the solar system from time to time through interstellar gas and dust.

In the present paper it is proposed to investigate solely the effects of passing stars on the orbits of the planet. A similar problem has been studied by Walters<sup>(1)</sup> for the case of spectroscopic binaries with the aim of explaining the observed correlations between the period and the eccentricity. He obtained expressions for the changes of the orbital elements of a binary system and calculated the average values of the variations assuming that the velocities of passing stars are distributed at random.

For any dynamically bound system such as the solar system or a binary star, an encounter with a passing star must clearly be regarded as a purely random process, and the effects of such random processes can be treated only statistically. Thus, although the average values are important for the description of the processes investigated, they do not give enough information of the processes. Walters's treatment is not satisfactory in this sense. Moreover, he took into account only the first order terms in the series expansions of the variations of the elements, whereas it will be shown here that for the planets much greater contributions to the variations of the semi-major axis and the eccentricity come from terms that beforehand would be expected to be ordinary second-order terms.

The problem of the effects of stellar encounters is clearly of importance when one considers the dynamical stability of the solar system. In the past the stability has been investigated on the basis that the planets and sun form an autonomous system, but it is necessary to know whether or not encounters have important effects, since if they do the usual treatment would have to be modified considerably.

In the present discussion, expressions for the changes in the orbital elements due to a single encounter in terms of the orbital elements of the passing star will be obtained by the method of variation of elements. We have essentially a restricted three-body problem in which the mass of the planet is so small that the relative motion of the two massive bodies, the sun and the passing star, is not disturbed by the planet, and may therefore be regarded as accurately hyperbolic.

Assuming that the velocities of the passing stars are distributed at random, the probability distributions of the variations of the elements will then be obtained by making use of the central-limit theorem of probability theory.

On the other hand, for very close encounters in which the star passes at planetary distances, the analytical methods here developed cannot be applied, and probably none are possible. For such cases the equations of motion have been integrated numerically.

2. Equations of Motion. - We adopt the following notation:

$M$  = mass of the sun

$M_1$  = mass of the passing star

$X_1, Y_1, Z_1$  : rectangular coordinates of the passing star with respect to non-rotating axes at the sun

$x, y, z$  : rectangular coordinates of the planet with respect to the sun

$\gamma$  = constant of gravitation ( $6.668 \times 10^{-8}$  c.g.s.).

We shall write  $\underline{X}_1 = (X_1, Y_1, Z_1)$ ,  $\underline{x} = (x, y, z)$ .

On the assumption that the mass of the planet is negligibly small compared with  $M$  and  $M_1$ , the equations of motion are:

$$\frac{d^2}{dt^2} (X_1) = - \gamma (M + M_1) \frac{X_1}{|X_1|^3}, \quad (2.1)$$

$$\frac{d^2}{dt^2} (x) + \gamma M \frac{x}{|x|^3} = M_1 \frac{X_1 - x}{|X_1 - x|^3} - M_1 \frac{X_1}{|X_1|^3}, \quad (2.2)$$

together with similar equations relating to  $Y_1, Z_1$ , and  $y, z$ . Equation (2.1) relates to the motion of the passing star relative to the sun which is hyperbolic in our problem. Equation (2.2) relates to perturbed elliptic motion of the planet round the sun under a disturbing function  $\Omega$  given by

$$\Omega \equiv \gamma M_1 \frac{1}{|X_1 - x|} - \gamma M_1 \frac{X_1 \cdot x}{|X_1|^3} \quad (2.3)$$

The  $(x, y)$  plane is now defined as the orbital plane of the passing star, and the hyperbolic motion of the star is thus determined by the following set of equations (2):

$$\begin{aligned} X_1 &= R_1 \cos v_1 = a_1(e_1 - \cosh D), \\ Y_1 &= R_1 \sin v_1 = a_1(e_1^2 - 1)^{\frac{1}{2}} \sinh D, \\ R_1 &\equiv (X_1^2 + Y_1^2)^{\frac{1}{2}} = a_1(e_1 \cosh D - 1), \\ n_1 t + \chi_1 &= e_1 \sinh D - D, \text{ where } n_1^2 a_1^3 = \gamma (M + M_1), \\ \cos v_1 &= \frac{e_1 - \cosh D}{e_1 \cosh D - 1}, \quad \sin v_1 = \frac{(e_1^2 - 1)^{\frac{1}{2}} \sinh D}{e_1 \cosh D - 1}, \end{aligned} \quad (2.4)$$

where  $v_1$  is the true anomaly, and  $D$  is the analogue of the eccentric anomaly. We later choose the origin of time so that  $\chi_1 = 0$ .

For the planetary orbit we have

$$\begin{aligned} r &= a(1 - e \cos u), \\ u - e \sin u &= nt + \chi, \text{ where } n^2 a^3 = \gamma M, \\ \cos v &= \frac{\cos u - e}{1 - e \cos u}, \quad \sin v = \frac{(1 - e^2)^{\frac{1}{2}} \sin u}{1 - e \cos u}, \end{aligned} \quad (2.5)$$

where  $v$  is the true anomaly, and  $u$  is the eccentric anomaly.

Before proceeding, it is necessary to consider the general order of the numerical values that certain of the elements are likely to have. For the planets most have orbital eccentricities between 0 and 0.1. For the stars it is known that the mean velocity relative to the sun in the neighborhood of the solar system is  $20 \text{ km sec}^{-1}$ . We denote by  $q_1$  the shortest distance of approach of the star to the sun, and assume that  $q_1$  is some ten times the semi-major axis of the orbit of the planet. In hyperbolic motion, if  $V_\infty$  is the relative velocity of the star when at great distances, then

$$e_1 = \frac{V_\infty^2 q_1}{\gamma(M+M_1)} + 1 \quad (2.6)$$

Supposing  $M = M_1$ , it is found that  $e_1 \sim 10$  and  $e_1 \sim 10^2$ , if  $q_1 = 40 \text{ a.u.}$  and  $q_1 = 400 \text{ a.u.}$ , respectively. Thus our problem is such that the various quantities take values with the following orders of magnitude:

- (i) the eccentricity of the planetary orbit is of order  $10^{-1}$ ,
- (ii) the ratio of the semi-major axis  $a$  of the planetary orbit to the shortest distance  $q_1$  between the sun and the passing star is of order  $10^{-1}$ ,
- (iii) the eccentricity  $e_1$  of the hyperbolic orbit is of order 10 for inner planets and  $10^2$  for outer planets.

### 3. Expression for the Disturbing Function in Terms of the Orbital

Elements. - We denote by  $\theta$  the angle between the two vectors  $\underline{X}_1 = (X_1, Y_1, Z_1)$  and  $\underline{x} = (x, y, z)$ , and write

$$\cos \theta = S. \quad (3.1)$$

We shall assume that  $R_1$  remains throughout the encounter always much greater than  $r$ ; if  $q_1$  is sufficiently greater than the initial value of  $a$ , this

assumption will continue to hold. We are here concerned only with the passage of stars in paths lying strictly outside of the solar system, and it is not our intension to treat the case of a very close encounter at planetary distances by analytical methods.

The disturbing function  $\Omega$  can be expanded as usual in the following way as an absolutely convergent series:

$$\Omega = \gamma M_1 \left\{ \frac{1}{R_1} + \frac{r^2}{R_1^3} P_2(S) + \frac{r^3}{R_1^4} P_3(S) + \dots \right\}, \quad (3.2)$$

where  $P_n$  is the Legendre polynomial of the order  $n$ . Whence we have

$$R_1^2 \frac{\partial \Omega}{\partial r} = \gamma M_1 \left\{ \frac{2r}{R_1} P_2(S) + \frac{3r^2}{R_1^3} P_3(S) + \dots \right\} \quad (3.3)$$

$$R_1^2 \frac{\partial \Omega}{\partial R_1} = -\gamma M_1 \left\{ 1 + \frac{3r^2}{R_1^2} P_2(S) + \frac{4r^3}{R_1^3} P_3(S) + \dots \right\} \quad (3.4)$$

$$\frac{R_1^2}{r} \frac{\partial \Omega}{\partial S} = \gamma M_1 \left\{ \frac{r}{R_1} P_2'(S) + \frac{r^2}{R_1^2} P_3'(S) + \dots \right\} \quad (3.5)$$

where the dashes denote differentiation with respect to  $S$ .

In order to refer the orbit to the coordinate system fixed at the sun, we introduce the following quantities associated with the elliptic orbit:

$\emptyset$  = longitude of ascending node measured from x-axis,

$\omega$  = longitude of perihelion measured from the ascending node,

$i$  = inclination of the orbit measured from the (x,y)-plane.

We then have at once

$$S \cos \theta = \cos(\omega + v) \cos(v_1 - \emptyset) + \cos i \sin(\omega + v) \sin(v_1 - \emptyset) \quad (3.6)$$

4. Numerical Values of Certain Definite Integrals. - Before we proceed to the calculation of the variations of the orbital elements it is necessary to evaluate a number of definite integrals that are relevant in later sections. We define functions  $L$ ,  $M$  and  $N$  by

$$\begin{aligned}
 L &= \int_0^{\infty} \frac{\cos \Delta}{\cosh^2 D} dD, \\
 M &= \int_0^{\infty} \frac{\sin \Delta \cos v_1 \sin v_1}{\cosh^2 D} dD, \\
 N &= \int_0^{\infty} \frac{\cos \Delta \cos^2 v_1}{\cosh^2 D} dD,
 \end{aligned} \tag{4.1}$$

where  $\Delta$  is defined by

$$\Delta = \frac{n}{n_1} (e_1 \sinh D - D). \tag{4.2}$$

Also  $\sin v_1$  and  $\cos v_1$  here are expressed by (2.4) as functions of  $D$ , while  $n_1$  and  $n$  are defined in (2.4) and (2.5), respectively, and  $e_1$  is the eccentricity of the hyperbolic orbit. Since  $e_1$  is large, of the order of 10 or greater, we can approximate  $\Delta$  by

$$\Delta = A \sinh D, \quad \text{where } A = \frac{n}{n_1} e_1. \tag{4.3}$$

Replacing  $D$  into  $\xi$  where

$$\sinh D = \xi, \tag{4.4}$$

we get the following expression for  $L$  in terms of  $A$ :

$$L(A) = \int_0^{\infty} \frac{\cos A \xi}{(1 + \xi^2)^{\frac{3}{2}}} d\xi. \tag{4.5}$$

The integral (4.5) is seen to be uniformly convergent, for

$$\int_0^{\infty} \left| \frac{\cos A \xi}{(1 + \xi^2)^{\frac{3}{2}}} \right| d\xi \leq \int_0^{\infty} \frac{d\xi}{(1 + \xi^2)^{\frac{3}{2}}} = 1.$$

Therefore we have

$$\frac{d}{dA} L(A) = \int_0^{\infty} \frac{d}{dA} \frac{\cos A \xi}{(1 + \xi^2)^{\frac{3}{2}}} d\xi = - \int_0^{\infty} \frac{A \sin A \xi}{(1 + \xi^2)^{\frac{3}{2}}} d\xi \tag{4.6}$$

on integration by parts. Now we have (3)

$$\int_0^{\infty} \frac{\cos A \xi}{(1 + \xi^2)^{\frac{3}{2}}} d\xi = K_0(A), \quad (4.7)$$

where  $Kn(A)$  is the modified Bessel-function defined by

$$Kn(x) = \frac{\pi}{2} \lim_{\nu \rightarrow \infty} \frac{I_{-\nu}(A) - I_{\nu}(A)}{\sin \nu \pi},$$

$$I_{\nu}(A) = \sum_{m=0}^{\infty} \left(\frac{A}{2}\right)^{\nu+2m} \frac{1}{m! \Gamma(\nu+m+1)} \quad (4.8)$$

Thus we have

$$\frac{d}{dA} L(A) = -AK_0(A) = \frac{d}{dA} \left\{ AK_1(A) \right\},$$

so that

$$L(A) = AK_1(A) + c,$$

where  $c$  is a constant of integration. To determine  $c$  we put  $A = 0$  in  $L(A)$ , giving

$$L(0) = \int_0^{\infty} \frac{1}{(1 + \xi^2)^{\frac{3}{2}}} d\xi = 1.$$

Since the behavior of  $Kn(A)$  near  $A = 0$  is given by (4)

$$Kn(A) \sim \frac{1}{2} \Gamma(n) \left(\frac{2}{A}\right)^n \quad 0 < A \ll 1,$$

we finally get that  $c = 0$ , and hence that

$$L(A) = A K_1(A). \quad (4.9)$$

Similarly it may be shown that

$$M(A) = \frac{1}{3} A^2 K_1(A), \quad (4.10)$$

$$N(A) = \frac{1}{3} A^2 K_2(A) . \quad (4.11)$$

We can now make estimates of the values of L, M, and N. For this purpose we first notice that

$$A^2 = \frac{M}{M+M_1} \left( \frac{q_1}{a} \right)^3 \frac{1}{e_1} ,$$

and take as a numerical example

$$\frac{a}{q_1} = 10^{-1} , \quad e_1 = 10, \quad M = M_1 .$$

The A is found to be 7, and is sufficiently large compared with unity to permit the use of the asymptotic expansion for the Bessel-functions, (4) namely:

$$K_n(A) \sim \left( \frac{\pi}{2A} \right)^{\frac{1}{2}} e^{-A} \left\{ 1 - \frac{1-4n^2}{8A} + \dots \right\} .$$

In this way we find

$$L(7) \sim 3 \times 10^{-3} , \quad M(7) \sim 7 \times 10^{-3} , \quad N(7) \sim 7 \times 10^{-3} . \quad (4.12)$$

For a passing star with  $q_1 = 40$  a.u.,  $e_1$  is 10, and A is about 7 for  $q_1 = 10$  a. On the other hand, if  $a = 40$  a.u., and in order that A may be as large as 6 or 7,  $q_1$  will have to be some 25 a. We shall therefore consider passing stars with  $q_1$  greater than 10 a for inner-planets, and  $q_1$  greater than 25 a for outer-planets. Then we may regard the values of L, M, and N given by (4.12) as definitely indicating the upper limits of  $L(A)$ ,  $M(A)$ , and  $N(A)$ .

As will be shown in later sections, the first order terms of the series expansions for the changes in the semi-major axis and the eccentricity contain L, M and N as multiplicative factors, and since L, M and N are small quantities of order  $10^{-2}$ , the main contributions come from the second order

terms. On the other hand if we consider a visual binary star with  $a$  as large as  $10^3$  a.u. and a passing star with  $q_1 = 10^4$  a.u., it is then found that  $A = 0.4$ , and hence  $L$ ,  $M$  and  $N$  are of order 1. Thus for such binaries the first order terms definitely give the largest contributions. In this respect Pluto may be regarded as intermediate between closely bound inner-planets and widely separated binaries.

5. Variation of Semi-major Axis. - We adopt the usual form of the equations for the variations of elements as given by Smart (5), since not only is this form suitable for the problem in hand but is also the form adopted by Walters, thus making the comparison of the present treatment with that of Walters more direct.

The equation for the variation of  $a$  is given by

$$\frac{da}{dt} = \frac{2}{na} \frac{\partial \Omega}{\partial x} . \quad (5.1)$$

The right hand side of the above equation can be expressed in terms of the orbital elements and the time, and in the first approximation the orbital elements that appear on the right may be regarded as constants in the standard way. The disturbing function  $\Omega$ , when expressed in the form (3.2), is a function of  $r$  and  $S$ , and we have

$$\frac{\partial \Omega}{\partial x} = \frac{\partial \Omega}{\partial r} \frac{a^2}{r} e \sin u + \frac{\partial \Omega}{\partial S} \frac{\partial S}{\partial v} \frac{\partial v}{\partial x} . \quad (5.2)$$

From (2.5) we get

$$\frac{\partial u}{\partial x} = \frac{1}{1-e \cos u} = \frac{a}{r} , \quad \frac{\partial v}{\partial u} = \frac{(1-e^2)^{\frac{1}{2}}}{1-e \cos u} = \frac{a(1-e^2)^{\frac{1}{2}}}{r} , \quad (5.3, 5.4)$$

and from (3.6)

$$\frac{\partial S}{\partial v} = - \sin(\omega+v) \cos(v_1-\phi) + \cos i \cos(\omega+v) \sin(v_1-\phi) . \quad (5.5)$$

Thus (5.1) may be written as

$$\frac{da}{dt} = \frac{2\gamma M_1}{na} \left\{ ae \sin u \left[ \frac{2a}{R_1^3} P_2(S) + \frac{3ar}{R_1^4} P_3(S) + \dots \right] + \frac{\partial S}{\partial v} (1-e^2)^{\frac{1}{2}} \left[ \frac{a^2}{R_1^3} P_2'(S) + \frac{a^2 r}{R_1^4} P_3'(S) + \dots \right] \right\}. \quad (5.6)$$

We have adopted the time-origin such that  $x_1 = 0$ , and we also neglect the terms of order  $e_1^{-1}$ . Then Eq. (2.4) reduces to

$$\begin{aligned} R_1 &= a_1 e_1 \cosh D, & n, t &= e_1 \sinh D, \\ \cos v_1 &= \operatorname{sech} D, & \sin v_1 &= \tanh D. \end{aligned} \quad (5.7)$$

The next step is to integrate (5.6) with respect to  $t$  from  $-\infty$  to  $+\infty$ . In his treatment Walters neglected all the terms in (5.6), except the apparently largest term, which is the first term in the second bracket, and then took the average value of this term due to stellar encounters. However, the average value of this term is zero, and in order to get non-zero contributions he made the invalid step of proceeding simply to the second approximation. It is necessary first, to consider each term in (5.6) respectively. For economy of writing we introduce the following notations and definitions:

$$\begin{aligned} I_1 &= 2a^2 e \int_{-\infty}^{\infty} R_1^{-3} \sin u P_2(S) dt, \\ I_2 &= 3a^2 e \int_{-\infty}^{\infty} r R_1^{-4} \sin u P_3(S) dt, \\ I_3 &= a^2 \int_{-\infty}^{\infty} R_1^{-3} \frac{\partial S}{\partial v} P_2'(S) dt, \\ I_4 &= a^2 \int_{-\infty}^{\infty} r R_1^{-4} \frac{\partial S}{\partial v} P_3'(S) dt. \end{aligned} \quad (5.8)$$

The approximate value of the change of the semi-major axis owing to the action of a star during a single passage is then given by

$$\delta a = \frac{2\gamma M_1}{na} (I_1 + I_2 + I_3 + I_4). \quad (5.9)$$

First we consider  $I_1$ . Let us write, for brevity,

$$\begin{aligned}\alpha &\equiv \cos\omega \cos(v_1 - \emptyset) + \cos i \sin\omega \sin(v_1 - \emptyset), \\ \beta &\equiv -\sin\omega \cos(v_1 - \emptyset) + \cos i \cos\omega \sin(v_1 - \emptyset),\end{aligned}\tag{5.10}$$

so that by (3.6)

$$S = \alpha \cos v + \beta \sin v.\tag{5.11}$$

Writing now for the mean anomaly

$$K = nt + \chi\tag{5.12}$$

the true anomaly  $v$  and the eccentric anomaly  $u$  can be expressed in terms of  $k$  by the following formulae (5), in which  $J_p$  is the Bessel-function of order  $p$ :

$$\begin{aligned}\sin u &= \frac{2}{e} \sum_{p=1}^{\infty} \frac{\sin pk}{p} J_p(pe) \\ &= (1 - \frac{1}{8} e^2) \sin k + \frac{1}{2} e \sin 2k + \frac{3}{8} e^2 \sin 3k + \dots, \\ \cos u &= -\frac{1}{2} e + 2 \sum_{p=1}^{\infty} \frac{\cos pk}{p} J'_p(pe) \\ &= (1 - \frac{3}{8} e^2) \cos k - \frac{1}{2} e + (\frac{1}{2} e - \frac{1}{3} e^3) \cos 2k + \frac{3}{8} e^2 \cos 3k + \dots, \\ \sin v &= 2(1-e^2)^{\frac{1}{2}} \sum_{p=1}^{\infty} J'_p(pe) \sin pk \\ &= (1-e^2)^{\frac{1}{2}} (1 - \frac{3}{8} e^2) \sin k + e \sin 2k + \frac{9}{8} e^2 \sin 3k + \dots \\ \cos v &= -e + \frac{2(1-e^2)}{e} \sum_{p=1}^{\infty} J_p(pe) \cos pk \\ &= (1-e^2) \cos k = e + e \cos 2k + \frac{9}{8} e^2 \cos 3k + \dots.\end{aligned}\tag{5.13}$$

to the second order in  $e$ . Expressing  $P_2(S)$  in terms of  $v$  by (5.11), the

$l_1$  separates into four parts, thus:

$$l_1 = (l_{1,1} + l_{1,2} + l_{1,3} + l_{1,4}) e,$$

where

$$\begin{aligned} l_{1,1} &= 3a^2 \int_{-\infty}^{\infty} R_1^{-3} \alpha^2 \sin u \cos^2 v \, dt, \\ l_{1,2} &= 3a^2 \int_{-\infty}^{\infty} R_1^{-3} \beta^2 \sin u \sin^2 v \, dt, \\ l_{1,3} &= 6a^2 \int_{-\infty}^{\infty} R_1^{-3} \alpha \beta \sin u \sin v \cos v \, dt, \\ l_{1,4} &= a^2 \int_{-\infty}^{\infty} R_1^{-3} \sin u \, dt. \end{aligned} \quad (5.14)$$

We consider  $l_{1,2}$  first.

$$\sin^2 v = \frac{1}{2} (1 - \cos 2k) + e (\cos k - \cos 3k) + \text{terms in } e^2 \text{ and higher.}$$

Therefore as long as the terms of small quantities of order  $e^2$  and higher are neglected, we have

$$\sin u \sin^2 v = \text{linear sum of } \sin k, \sin 2k, \sin 3k, \sin 4k. \quad (5.15)$$

From (5.7) we have

$$dt = \frac{e_1}{n_1} \cosh d \, dd. \quad (5.16)$$

Thus, when (5.15) is inserted in (5.14) together with (5.16) for  $dt$  and (5.7) for  $R_1$ ,  $l_{1,2}$  is expressed in terms of  $L(A)$ ,  $M(A)$ , and  $N(A)$ , and since  $L(A)$ ,  $M(A)$  and  $N(A)$  are of order  $10^{-2}$ , as we have seen in section 4,  $l_{1,2}$  is of the order of  $10^{-2}$ . In a similar way it can be shown that  $l_{1,1}$  and  $l_{1,4}$  can be expressed in terms of  $L(A)$ ,  $M(A)$ , and  $N(A)$  as far as the terms of order  $e^2$  and higher are neglected, and therefore they are of order  $10^{-2}$ .

Next we consider  $l_{1,3}$ . We have

$$\sin u \sin v \cos v = -\frac{3}{8} e + \cos k \sin^2 k + e(\cos 2k - \frac{5}{8} \cos 4k) + O(e^2),$$

and when this is inserted in (5.14), all the terms except the first can be expressed in terms of  $L(A)$ ,  $M(A)$ , and  $N(A)$  together with similar ones in which  $A$  is replaced by  $2A$ ,  $3A$ , etc. Since the first term of  $l_{1,3}$  is of order  $e$ , whereas  $l_{1,1}$ ,  $l_{1,2}$ , and  $l_{1,4}$  are of order  $10^{-2}$ , the largest contribution to  $l_1$  comes from  $l_{1,3}$ .

For the integration of  $l_{1,3}$  we need the following definite integrals:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\cos^2(v_1 - \vartheta)}{\cosh^2 D} dD &= \frac{4}{3} \cos^2 \vartheta + \frac{2}{3} \sin^2 \vartheta \\ \int_{-\infty}^{\infty} \frac{\cos(v_1 - \vartheta) \sin(v_1 - \vartheta)}{\cosh^2 D} dD &= -\frac{2}{3} \cos \vartheta \sin \vartheta \\ \int_{-\infty}^{\infty} \frac{\sin^2(v_1 - \vartheta)}{\cosh^2 D} dD &= \frac{4}{3} \sin^2 \vartheta + \frac{2}{3} \cos^2 \vartheta \end{aligned} \quad (5.17)$$

Thus after a little algebra we finally obtain

$$\begin{aligned} l_1 = & -\frac{9a^2 e^2}{4a_1^3 e_1^2 n_1} \left\{ \left( -\frac{4}{3} \cos^2 \vartheta + \frac{2}{3} \sin^2 \vartheta \right) \cos \omega \sin \omega \right. \\ & + \cos^2 i \sin \omega \cos \omega \left( \frac{4}{3} \sin^2 \vartheta + \frac{2}{3} \cos^2 \vartheta \right) + \cos i (\cos^2 \omega - \sin^2 \omega) \times \\ & \left. \left( -\frac{2}{3} \cos \vartheta \sin \vartheta \right) \right\}. \end{aligned} \quad (5.18)$$

To evaluate  $l_2$ , given by (5.8), we have to begin with

$$\begin{aligned} \frac{r}{a} \sin u P_3(S) &= (1-e \cos u) \sin u S \left( \frac{5}{2} S^2 - \frac{3}{2} \right) \\ &= \frac{5}{2} \left\{ \left[ \frac{1}{2} \cos^2(v_1 - \vartheta) - \frac{1}{2} \cos^2 i \sin^2(v_1 - \vartheta) \right] \left[ -\frac{1}{4} \sin \omega \cos(v_1 - \vartheta) \right. \right. \\ &\quad \left. \left. - \frac{1}{4} \cos \omega \cos i \sin(v_1 - \vartheta) \right] + \left[ \frac{1}{2} \cos^2(v_1 - \vartheta) + \frac{1}{2} \cos^2 i \sin^2(v_1 - \vartheta) \right] \times \right. \\ &\quad \left. \left[ -\frac{1}{2} \sin \omega \cos(v_1 - \vartheta) + \frac{1}{2} \cos \omega \cos i \sin(v_1 - \vartheta) \right] \right. \\ &\quad \left. + \cos i \cos(v_1 - \vartheta) \sin(v_1 - \vartheta) \left[ \frac{1}{4} \cos \omega \cos(v_1 - \vartheta) - \frac{1}{4} \sin \omega \cos i \sin(v_1 - \vartheta) \right] \right\} \\ &+ \text{terms in } \sin 2k, \cos 2k, \\ &+ \text{terms of order } e \text{ and higher.} \end{aligned} \quad (5.19)$$

The terms containing  $\sin 2k$ ,  $\cos 2k$ , etc. can be expressed in terms of the integral

$$H(2A) \equiv \int_0^\infty \frac{\cos 2A \xi}{(1 + \xi^2)^2} d\xi = \frac{(1+2A)}{4} e^{-2A} \sim 10^{-4}.$$

Therefore the terms that contain  $\sin 2k$ ,  $\cos 2k$ , and terms of order  $e$  and higher may be neglected compared with the rest which involve neither  $e$  nor  $\sin 2k$ ,  $\cos 2k$ , etc. For the integration of  $I_2$  we need the following integrals:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\cos^3(v_1 - \vartheta)}{\cosh^3 \vartheta} d\vartheta &= \frac{16}{15} \cos^3 \vartheta + \frac{12}{15} \sin^2 \vartheta \cos \vartheta \\ \int_{-\infty}^{\infty} \frac{\cos^2(v_1 - \vartheta) \sin(v_1 - \vartheta)}{\cosh^3 \vartheta} d\vartheta &= \frac{4}{15} \cos^3 \vartheta + \frac{8}{15} \cos^2 \vartheta \sin^2 \vartheta \\ \int_{-\infty}^{\infty} \frac{\sin^2(v_1 - \vartheta) \cos(v_1 - \vartheta)}{\cosh^3 \vartheta} d\vartheta &= \frac{4}{15} \cos^3 \vartheta + \frac{8}{15} \cos \vartheta \sin^2 \vartheta \\ \int_{-\infty}^{\infty} \frac{\sin^3(v_1 - \vartheta)}{\cosh^3 \vartheta} d\vartheta &= -\frac{16}{15} \sin^3 \vartheta - \frac{12}{15} \sin \vartheta \cos^2 \vartheta \end{aligned} \quad (5.20)$$

We do not, however, express  $I_2$  explicitly until we come to the calculation of  $I_4$ .

Next we consider  $I_3$ :

$$\begin{aligned} P_2'(S) \frac{\partial S}{\partial v} &= -\frac{3}{2} \sin 2(\omega + v) [\cos^2(v_1 - \vartheta) - \cos^2 i \sin(v_1 - \vartheta)] \\ &+ 3 \cos i \cos 2(\omega + v) \cos(v_1 - \vartheta) \sin(v_1 - \vartheta). \end{aligned}$$

It can be shown that

$$\sin 2v = \text{linear sum of } \sin 2k, \sin 3k, + \text{ terms of order } e^3 \text{ and higher,}$$

$$\begin{aligned} \cos 2v &= \frac{3}{4} e^2 + \text{linear combination of } e \cos k, \cos 2k, \dots \\ &+ \text{terms of } e^3 \text{ and higher.} \end{aligned}$$

Since the terms of  $\sin 2k$ ,  $\sin 3k$ ,  $e \cos k$ , etc. will give contributions less than  $10^{-3}$ , the largest contribution to  $I_3$  comes from the term  $3e^2/4$  in  $\cos 2v$ , which is of order  $10^{-2}$ . After integration we get

$$I_3 = \frac{a^2 e^2}{a_1^3 e_1^2 n_1} \left\{ -\frac{9}{8} \sin 2\omega \left[ \left( \frac{4}{3} \cos^2 \theta + \frac{2}{3} \sin^2 \theta \right) - \cos^2 i \left( \frac{4}{3} \sin^2 \theta + \frac{2}{3} \cos^2 \theta \right) \right] \right. \\ \left. - \frac{3}{2} \cos i \cos 2\omega \cos \theta \sin \theta \right\}. \quad (5.21)$$

We next consider  $I_4$ :

$$\frac{r}{a} \frac{\partial S}{\partial v} P_3'(S) = (1-e \cos u) \frac{\partial S}{\partial v} P_3'(S) \\ = -\frac{15}{2} e \left\{ \frac{1}{2} \left[ \cos^2(v_1 - \theta) - \cos^2 i \sin^2(v_1 - \theta) \right] \times \right. \\ \left[ \frac{1}{4} \sin \omega \cos(v_1 - \theta) + \frac{1}{4} \cos \omega \cos i \sin(v_1 - \theta) \right] \\ + \frac{1}{2} \left[ \cos^2(v_1 - \theta) + \cos^2 i \sin^2(v_1 - \theta) \right] \times \\ \left[ -\frac{1}{2} \sin \omega \cos(v_1 - \theta) + \frac{1}{2} \cos \omega \cos i \sin(v_1 - \theta) \right] \\ + \cos i \cos(v_1 - \theta) \sin(v_1 - \theta) \left[ -\frac{1}{4} \cos(v_1 - \theta) \cos \omega + \frac{1}{4} \cos i \sin(v_1 - \theta) \sin \omega \right] \Big\} \\ + \frac{3}{2} e \left[ -\frac{1}{2} \sin \omega \cos(v_1 - \theta) + \frac{1}{2} \cos i \cos \omega \sin(v_1 - \theta) \right] \\ + \frac{15e}{8} \left\{ -\sin^3 \omega \cos^3(v_1 - \theta) + \cos^3 i \cos^3 \omega \sin^3(v_1 - \theta) \right. \\ + 3 \sin^2 \omega \cos^2(v_1 - \theta) \cos i \cos \omega \sin(v_1 - \theta) \\ - 3 \sin \omega \cos(v_1 - \theta) \cos^2 i \cos^2 \omega \sin^2(v_1 - \theta) \Big\} \\ - \frac{15e}{8} \left\{ -\sin \omega \cos^2 \omega \cos^3(v_1 - \theta) + \cos^3 i \sin^2 \omega \cos \omega \sin^3(v_1 - \theta) \right. \\ + \cos^2(v_1 - \theta) \sin(v_1 - \theta) + \cos i \left[ \cos^3 \omega - 2 \cos \omega \sin^2 \omega \right] \\ + \cos(v_1 - \theta) \sin^2(v_1 - \theta) \cos^2 i \left[ 2 \cos^2 \omega \sin \omega - \sin^3 \omega \right] \\ + \frac{9e}{4} \left[ -\sin \omega \cos(v_1 - \theta) + \cos i \cos \omega \sin(v_1 - \theta) \right] \Big\} \\ + \text{terms in } e^2 \text{ and higher, and the terms } \sin k, \cos k. \quad (5.22)$$

Neglecting the terms in  $\sin k$ ,  $\cos k$ , which are of order  $10^{-2}$ , and the terms of order  $e^2$  and higher, we can integrate (5.22) with respect to  $t$  from  $-\infty$  to  $+\infty$  by making use of the integrals of (5.20). Then it is found that the sum of  $l_2$  and  $l_4$  may be put into the following form:

$$l_2 + l_4 = \frac{a^3 e}{a_1^4 e_1^3 n_1} F, \quad (5.23)$$

where  $F$  depends only on the angles  $\varphi$ ,  $\omega$ ,  $i$ , and is given by

$$F = \sin \omega [\cos^3 \varphi - \cos^2 i (\cos^3 \varphi + 2 \cos \varphi \sin^2 \varphi)] \\ - \cos i \cos \omega \left[ \frac{1}{2} \sin \varphi \sin^2 i - \sin^3 \varphi \sin^2 i - 2 \cos^2 \varphi \sin \varphi \right]. \quad (5.24)$$

The contributions from  $l_1$  and  $l_3$  cancel each other. Thus finally we get the following expression for the change of the semi-major axis due to a passing star:

$$\frac{na}{2} \frac{\delta a}{M_1} = \frac{a^3 e}{a_1^4 e_1^3 n_1} F, \quad (5.25)$$

where  $F$  is given by (5.24). This expression can be put into a more convenient one as follows:

$$\frac{\delta a}{ae} = \frac{2M_1}{M^2 (M+M_1)^2} \left( \frac{a}{q_1} \right)^{\frac{5}{2}} \frac{F}{e_1^{\frac{1}{2}}}, \quad (5.26)$$

where  $q_1$  is the shortest distance between the sun and the passing star. To estimate the value of  $\delta a/a$  we consider an example such that  $q_1 = 10a$  and  $e_1 = 10$ . This corresponds to a passing star of initial relative velocity  $20 \text{ km. sec}^{-1}$ . disturbing a planet with  $a = 4 \text{ a.u.}$  Since  $F$ , given by (5.24) is of order 1, we get

$$\frac{\delta a}{ae} \sim 1.4 \times 10^{-3}.$$

On the other hand if we consider Pluto and a passing star of  $q_1 = 1000$  a.u. so that  $q_1 = 25a$  and  $e_1 = 250$ , it is then found that

$$\frac{\delta a}{ae} \sim 3 \times 10^{-5}.$$

It may be noticed that, by (2.6),  $e_1$  is proportional to  $q_1$  and therefore  $\delta a/a$  is inversely proportional to  $q_1^3$ . Again since  $e_1$  is proportional to  $V_\infty^2$ , the variation of  $a$  is inversely proportional to  $V_\infty$ , as might be expected.

In his treatment Walters considers  $l_3$  over uniformly distributed  $\phi, \omega, i$  is zero, then proceeds to the second approximation where the elements that appear on the right-hand side of (5.1) are not regarded as constant but variable with time. Again he calculates the integral

$$\int_0^\infty \frac{\cos A\xi}{(1+\xi^2)^{\frac{3}{2}}} d\xi,$$

and other similar integrals by numerical quadratures. But these quantities are in fact expressible in terms of Bessel-functions, and as we have seen, are of order of magnitude  $10^{-2}$ .

It is necessary to consider the accuracy of expression (5.26). The terms in (5.6) which contain  $P_4(S)$  or  $P_4'(S)$  are easily shown to be expressible in terms of  $\sin k$ ,  $\cos k$ , and terms of order  $e$  or higher. Thus these terms are definitely smaller than  $l_2$  or  $l_4$  by a factor of  $10^{-1}$ . In our calculations small quantities of relative magnitude  $10^{-1}$  have systematically been neglected. Thus we may conclude that the accuracy is of relative error  $10^{-1}$ .

For an extremely high velocity star with, say,  $V_\infty = 200$  km. sec $^{-1}$ ,  $A$  defined by (4.3) will be very small, and in fact,  $A = 0.2$  if  $q_1 = 400$  a.u. and  $a = 40$  a.u. For such small values of  $A$ ,  $L$ ,  $M$ , and  $N$  are of order 1, and the main contribution to the change in  $a$  will come from  $l_3$  which is the first order term in (5.6). Although it is not the purpose of the present paper to discuss such extreme cases in detail it will still be of interest to obtain the order

of magnitude in  $\delta a/a$ . It can easily be shown that, after integration of  $I_3$  term in (5.6),

$$\frac{\delta a}{a} = \frac{2M_1}{M^2(M+M_1)^2} \left(\frac{a}{q_1}\right)^{\frac{3}{2}} \frac{1}{e_1^2} \left\{ \text{linear sum of } L(2A), M(2A), \text{ and } N(2A) \right\},$$

and this is of the order of  $10^{-4}$ . For widely separated binaries  $A$  is again small even for a passing star of moderate velocity, and the change in  $a$  will again take the above form. This problem will, however, not be discussed in the present paper.

6. Variation of Eccentricity. - The eccentricity changes according to the equation

$$\frac{de}{dt} = \frac{1-e^2}{na^2e} \frac{\partial \Omega}{\partial x} - \frac{(1-e^2)^{\frac{1}{2}}}{na^2e} \frac{\partial \Omega}{\partial \omega}. \quad (6.1)$$

Using (5.3) and (5.4), together with the identity

$$\frac{\partial S}{\partial \omega} \equiv \frac{\partial S}{\partial v},$$

we get

$$\begin{aligned} \frac{na}{M_1} \frac{de}{dt} = \frac{(1-e^2)^{\frac{1}{2}}}{1-e \cos u} \left\{ (1-e^2)^{\frac{1}{2}} \sin u \left[ \frac{2r}{R_1^3} P_2(S) + \frac{3r^2}{R_1^4} P_3(S) + \dots \right] \right. \\ \left. + \frac{2 \cos u - e(1+\cos^2 u)}{r} \left[ \frac{r^2}{R_1^3} P_2'(S) + \frac{r^3}{R_1^4} P_3'(S) + \dots \right] \frac{\partial S}{\partial v} \right\}. \end{aligned} \quad (6.2)$$

The calculation can be carried out in a similar manner to that of section 5.

The initial terms in the first and the second brackets give a contribution of order  $e$ , and the second terms in the first and the second brackets contribute by an amount of order  $a/q_1$ . The result of the calculation is

$$\frac{na \delta e}{M_1} = \frac{a}{a_1^3 e_1^2 n_1} \left( \frac{a}{a_1 e_1} G + eH \right), \quad (6.3)$$

where

$$G = \sin (\cos^3 \vartheta + \frac{3}{4} \sin^2 \vartheta \cos \vartheta + 2 \cos \vartheta) + \cos i \cos \omega (\frac{1}{4} \sin^3 \vartheta + \frac{1}{2} \cos^2 \vartheta \sin \vartheta) \\ + \cos^2 i \sin \omega (\frac{1}{4} \cos^3 \vartheta + \frac{1}{2} \cos \vartheta \sin^2 \vartheta) + \cos^3 i \cos \omega (\sin^3 \vartheta + \frac{3}{4} \sin \vartheta \cos^2 \vartheta) \quad (6.4)$$

$$H = - \frac{15}{2} \left\{ \cos \omega \sin \omega \left[ - \frac{4}{3} \cos^2 \vartheta - \frac{2}{3} \sin^2 \vartheta + \cos^2 i \left( \frac{4}{3} \sin^2 \vartheta + \frac{2}{3} \cos^2 \vartheta \right) \right] \right. \\ \left. - \frac{2}{3} \cos i \cos \omega \cos \vartheta \sin \vartheta \right\} \quad (6.5)$$

In (6.3) the terms involving  $P_4(S)$  and  $P_4'(S)$  in (6.2) are neglected. These terms contribute to  $\delta e$  by an amount that is less than  $\delta e$  as found above by a factor  $10^{-1}$ .

7. Variation of the Inclination. - The inclination varies according to the equations

$$\frac{di}{dt} = \frac{\cot i}{na^2(1-e^2)^{\frac{1}{2}}} \frac{\partial \Omega}{\partial \omega} - \frac{\operatorname{cosec} i}{na^2(1-e^2)^{\frac{1}{2}}} \frac{\partial \Omega}{\partial \phi} \quad (7.1)$$

$$= \frac{1}{na^2(1-e^2)^{\frac{1}{2}}} \frac{1}{\sin i} \frac{\partial \Omega}{\partial S} \left( \cos i \frac{\partial S}{\partial \omega} - \frac{\partial S}{\partial \phi} \right). \quad (7.2)$$

From (3.5) we get

$$\cos i \frac{\partial S}{\partial \omega} - \frac{\partial S}{\partial \phi} = \sin^2 i \cos (\omega + v) \sin(\vartheta - v_1).$$

Thus we obtain

$$\frac{di}{dt} = \frac{1}{na^2(1-e^2)^{\frac{1}{2}}} \sin i \cos(\omega + v) \sin(\vartheta - v_1) \frac{\partial \Omega}{\partial S}. \quad (7.3)$$

Using (3.4) we get, to order  $r^3/R_1^4$ ,

$$\cos(\omega + v) \sin(\vartheta - v_1) \frac{\partial \Omega}{\partial S} = M_1 (K_1 + K_2 + \dots),$$

where

$$K_1 = \cos(\omega + v) \sin(\vartheta - v_1) \frac{r^2}{R_1^3} P_2^1(S)$$

$$K_2 = \cos(\omega + v) \sin(\vartheta - v_1) \frac{r^3}{R_1^4} P_3^1(S) .$$

It may be shown that

$$K_1 = \frac{3}{2} \frac{r^2}{R_1^3} \sin(\vartheta - v_1) \cos(v_1 - \vartheta) + \text{terms in } \sin 2K, \cos 2K,$$

and terms of order  $e$  or higher,

and therefore even if we neglect  $K_2, \dots$  etc., the relative error is of the order of  $10^{-1}$ . Carrying out the integration we get

$$\delta i = \frac{M_1}{na^2(1-e^2)^{\frac{1}{2}}} \frac{a^3}{a_1^3 e_1^3} \frac{e_1}{n_1} \sin i \sin \vartheta \cos \vartheta . \quad (7.4)$$

This can be written more conveniently as;

$$\delta i = \frac{M_1}{M^2(M+M_1)^{\frac{1}{2}}} \frac{1}{e_1^{\frac{1}{2}}} \left(\frac{a}{q_1}\right)^{\frac{3}{2}} \sin i \sin \vartheta \cos \vartheta . \quad (7.5)$$

If  $q_1 = 1000$  a.u.,  $a = 40$  a.u. and is replaced by its maximum value 0.5 the approximate value of  $\delta i$  is found to be  $2 \times 10^{-4}$ . If on the other hand,

$q_1 = 40$  a.u. and  $a = 4$  a.u., we get

$$\delta i \sim 4 \times 10^{-3} .$$

8. Statistical Treatment. - The changes of the orbital elements of a planet are now available expressed in terms of the orbital elements of the passing star. In order to estimate the cumulative effect of successive variations of the elements, we have to consider the distribution of orbital elements of passing stars. At present the velocity distribution near the sun is believed to be ellipsoidal, but here we consider only an ideal

spherical distribution. Each passing star is assumed to have the same mass as the sun, and the velocities are distributed according to the Gaussian distribution

$$P(V) = \frac{4\alpha^3}{\pi^{\frac{3}{2}}} V^2 e^{-\alpha^2 V^2} dV, \quad (8.1)$$

where  $\alpha$  can be determined by the requirement that the average velocity must coincide with the average observed velocity. Since the average value of the observed velocity is  $20 \text{ km. sec}^{-1}$  we have

$$V_m \text{ (mean velocity)} = \frac{2}{(\pi)^{\frac{1}{2}}}, \quad (8.2)$$

but the actual numerical value of  $\alpha$  is not needed till the final stage.

For a spherical distribution of velocities, the probability distribution of  $i$  is proportional to  $\sin i$  so that

$$P(i) di = \frac{1}{2} \sin i di, \quad 0 \leq i < \pi,$$

and for the same reason,

$$P(\omega) d\omega = \frac{1}{2\pi} d\omega, \quad 0 \leq \omega < 2\pi,$$

$$P(\phi) d\phi = \frac{1}{2\pi} d\phi, \quad 0 \leq \phi < 2\pi.$$

In order to estimate the cumulative effect of the variations of the elements, we shall use the central-limit theorem of probability theory (6), which is as follows:

If  $\zeta_1, \zeta_2, \dots, \zeta_N$  is a sequence of events with the same probability distribution, then the probability distribution of  $\zeta = \zeta_1 + \zeta_2 + \dots + \zeta_N$  approaches asymptotically the normal distribution

$$P(x < \zeta < x + dx) \sim \frac{1}{(2\pi)^{\frac{1}{2}} \sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} dx, \quad (8.4)$$

as  $N$  increases indefinitely, where

$m' = N$  times the average value of the  $\zeta$ s ,

$\sigma' = N^{\frac{1}{2}}$  times the standard - deviation of the  $\zeta$ s.

Since the average values of the variation of the elements are zero, the problem of finding the cumulative effect is reduced to finding the standard-deviations of the variations.

Defining the effective range  $R$  such that stars with  $q_1$  greater than  $R$  can be neglected, we have

$$P(q_1) dq_1 = \frac{2}{R^2} q_1 dq_1 . \quad (8.5)$$

Using the above distribution functions, we can calculate the standard-deviations of the variations of the elements. In doing so, we must remember that our approximation is valid only when  $q_1$  is much greater than the semi-major axis of the orbit of the planet. Then, neglecting the effect of stars with  $q_1$  less than  $q_\ell$ , we get, using (5.26), (6.3), and (7.5), for the standard-deviations,

$$\sigma'(e^{-1} a^{-\frac{2}{2}} \delta a) = 4 \left( \frac{M\gamma}{\pi} \right)^{\frac{1}{2}} \frac{1}{V_m R} \left[ \frac{1}{q_\ell^4} - \frac{1}{R^4} \right]^{\frac{1}{2}} (0.197619)^{\frac{1}{2}} \quad (8.6)$$

$$\sigma'(a^{-\frac{3}{2}} \delta e) = 2 \left( \frac{M\gamma}{\pi} \right)^{\frac{1}{2}} \frac{1}{V_m R} \left[ 3.00952 a^2 \left( \frac{1}{q_\ell^4} - \frac{1}{R^4} \right) + 10 e^2 \left( \frac{1}{q_\ell^2} - \frac{1}{R^2} \right) \right]^{\frac{1}{2}} \quad (8.7)$$

$$\sigma'(a^{-\frac{3}{2}} \delta i) = \left( \frac{2M\gamma}{3\pi} \right)^{\frac{1}{2}} \frac{1}{V_m R} \left[ \frac{1}{q_\ell^2} - \frac{1}{R^2} \right]^{\frac{1}{2}} . \quad (8.8)$$

It is necessary at this stage to calculate the number of stars that enter the domain of a sphere with radius  $R$ . Regarding the stars as molecules of a gas, it is a well-known result that the number of molecules that cross an area  $dS$  from one side to the other during a time period  $T$  is (7)

$\frac{1}{4} V_m n T dS$ , and here for a sphere of radius  $R$  the required number is

$$N = R^2 \rho V_m T, \quad (8.9)$$

where  $\rho$  is the density and  $V_m$  is the mean velocity. Thus the standard-deviations  $\sigma'$  of the asymptotic normal distribution (8.4) are

$$\sigma'(\epsilon^{-1} a^{-\frac{7}{2}} \delta a) = \frac{4(M \sqrt{T})^{\frac{1}{2}}}{V_m^{\frac{1}{2}}} \left[ \frac{1}{q_p^4} - \frac{1}{R^4} \right]^{\frac{1}{2}} (0.197619)^{\frac{1}{2}} \quad (8.10)$$

$$\sigma'(a^{-\frac{3}{2}} \delta e) = \frac{2(M \sqrt{T})^{\frac{1}{2}}}{V_m^{\frac{1}{2}}} \left[ 3.00952 a^2 \left( \frac{1}{q_p^4} - \frac{1}{R^4} \right) + 10 e^2 \left( \frac{1}{q_p^2} - \frac{1}{R^2} \right) \right]^{\frac{1}{2}} \quad (8.11)$$

$$\sigma'(a^{-\frac{3}{2}} \delta i) = \frac{(2M \sqrt{T})^{\frac{1}{2}}}{(3V_m)^{\frac{1}{2}}} \left[ \frac{1}{q_p^2} - \frac{1}{R^2} \right]^{\frac{1}{2}} \quad (8.12)$$

In the above expressions we can now make  $R \rightarrow \infty$ , and thus the somewhat arbitrarily defined effective range  $R$  does not enter into the final expressions for the standard deviations.

We now apply the results to a hypothetical planet moving at the mean distance of Pluto 40 a.u. For this planet  $q_p$  can not be less than 1000 a.u. for the approximation to be valid. If we adopt

$T = 4000$  million years,  $\rho = 0.1$  star 1 cubic parsec,  
and putting  $e = 0.1$ , we get

$$\sigma'(\delta i) = 1.3 \times 10^{-4}$$

$$\sigma'(\delta a/a) = 1.2 \times 10^{-6} \quad \text{for } \underline{a} = 40 \text{ a.u. and } q_p = 1000 \text{ a.u.}$$

$$\sigma'(\delta e) = 1.2 \times 10^{-4}$$

On the other hand, for an inner-planet of  $\underline{a} = 4$  a.u.,  $q_p$  may be taken as small as 40 a.u., and we get

$$\sigma'(\delta i) = 1.0 \times 10^{-4}$$

$$\sigma'(\delta a/a) = 7.3 \times 10^{-7} \quad \text{for } \underline{a} = 4 \text{ a.u. and } q_p = 40 \text{ a.u.}$$

$$\sigma'(\delta e) = 9.0 \times 10^{-5}$$

In the above example for  $\underline{a} = 4$  a.u., it is not suggested that a star will come as close as  $q_p = 40$  a.u. during the time period of 4000 million years,

but  $q_1$  should be regarded as simply indicating the minimum value of  $q_1$  for which the analytical methods are valid.

9. Discussion of the preceding results. - In the preceding sections the effect of stellar encounters on the planetary motion has been investigated in analytical form and it seems worthwhile to discuss the results obtained before we proceed to the numerical treatment of the problem.

It has been found that the variation of the elements of the planetary orbit by a stellar encounter can be expressed in analytical forms ((5.26), (6.3) and (7.5) ) within the accuracy of 70 percent. An interesting feature of these expressions is that the variation of the elements does not depend on  $x$ , the mean anomaly of the planetary orbit at  $t = 0$ . When we assume a Gaussian distribution of stellar velocities, the cumulative effect can be expressed by (8.10), (8.11) and (8.12). It is found that the standard-deviations of the cumulative effect are small, and it can be concluded that, exceptional events apart, the solar system is dynamically isolated from the stars for a time considerably longer than the probable age of the solar system. Thus in considerations of the dynamical stability of the solar system it is almost certainly valid that the effect of stellar encounters can be neglected.

In the above discussion, however, the stars that come within the range of 10 times the semi-major axis of the planet are not taken into account. Although any such close encounter by a star may be a rare event, the discussion will be completed by considering the consequences of such close encounters. The analytical difficulties involved are so great that direct numerical integrations appear to be the only feasible means of coping with such a problem, and in the following section we explain the numerical method actually adopted.

10. Integrations of the equations of motion for close encounters.

(i) Numerical procedure for the integration of Eq. (2.2).

The coordinates of the passing star are again given by (2.4), and the mass of the passing star is taken equal to that of the sun. We adopt the following units:

$\gamma \equiv$  Constant of gravitation = 1

$M \equiv$  mass of the sun = 1

$a \equiv$  semi-major axis of planetary orbit before the encounter = 1.

Since our main interest is in the orbital elements after the encounter, and not in the position of the planet at each instant of time, we can use  $D$  defined by (2.4) as independent variable in place of  $t$ . There is another reason for the choice of  $D$  as independent variable, for when the passing star is far from the sun the disturbance of the elliptic planetary motion is small and large integration steps are permissible, whereas when the passing star is close to the sun, the disturbance is great and smaller integration steps are required. The use of  $D$  automatically takes care of this feature. To integrate the equations of motion, the Runge-Kutta-Gill's method was adopted, and the quantities calculated at each step of the integrations are:

1) the coordinates and velocities of the planet with respect to the sun,

2) twice the total mechanical energy  $E$  per unit mass of the planet;

$$2E = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 - \frac{2}{r}, \quad \text{where } r = (x^2 + y^2 + z^2)^{\frac{1}{2}}. \quad (10.1)$$

Unfortunately in the present problem no integral exists equivalent to the Jacobi-integral, since such an integral would be useful to check the accuracy of the numerical integrations at each stage. In fact, the computation of the mechanical energy at each step of the integration was made

in order to obtain some indication of the accuracy of the integrations. When the passing star is far from the solar system, the disturbance due to it must be small and the mechanical energy  $E$  should remain practically constant. Moreover we can see when we should stop our integrations by noting the behaviour of  $E$ . Thus the criterion was adopted that the integration should be stopped when  $E$  remains constant within the relative accuracy  $10^{-4}$  over 30 integration steps. Calculation of the distance between the planet and the passing star is also required, for if the planet gets very close to the passing star, accuracy will be lost, and subsequent results will not be reliable. For this reason we adopted the criterion that, if the distance between the passing star and the planet becomes less than 0.1 in the above units, the calculation shall stop. In Table 1, this occurrence is referred to as "collision". The integration step  $\Delta D$  taken was 0.01 for all the cases computed, except the one for which the closest approach of the passing star to the sun,  $q_1 = 0.5$  for which  $\Delta D = 0.0090975$  was adopted. With this step-length of integration, the passing star moves from  $R_1$  (distance between the sun and the star) = 10 to the symmetrical position  $R_1 = 10$  in 800 steps of integration. The interval  $\Delta D = 0.01$  was chosen from error considerations. For nearly all cases, 800 steps of integration were enough to satisfy the criterion that  $E$  should remain constant for 30 steps at the end of the integrations. When the planetary orbit was strongly disturbed, however, it was necessary to carry out the integrations more than 800 steps and the accuracy is not so high as for other cases. Whether the planetary orbit is hyperbolic or elliptic after an encounter can be found from the sign of  $E$ . for we have positive values corresponding to a hyperbolic orbit (and escape from the sun), and negative values to an elliptic orbit. The

orbital elements after the encounter are given by

$$\begin{aligned} a &= -\frac{1}{2E}, \quad e^2 = 1 - \frac{h^2}{a} \quad \text{for elliptic orbits} \\ a &= \frac{1}{2E}, \quad e^2 = 1 + \frac{h^2}{a} \quad \text{for hyperbolic orbits} \end{aligned} \tag{10.2}$$

$$\cos i = h_z/h \text{ in all cases,}$$

where  $h$  and  $h_z$  are the magnitude of the total angular momentum and its  $z$ -component respectively.

(ii) Results of the numerical integrations.

It is necessary to specify precise initial conditions, and for this purpose the integrations were started when the distance between the sun and the passing star was 10 times the distance between the sun and the planet; this epoch was defined as  $t = 0$ . For all the cases computed, the eccentricity of the planetary orbit at this epoch was taken as zero. The main results are shown in Table 1, in which  $\psi$  is the longitude of the planet at  $t = 0$ , measured from the  $x$ -axis. The initial velocities of the planet are in the direction of increasing longitude except in the cases shown with an asterisk for which the initial velocity is in the direction of the decreasing longitude. It is interesting to investigate if the planetary orbit becomes retrograde due to a stellar encounter. For coplanar cases this can be seen by comparing the signs of  $h_z$  before and after the encounter, and for none of the computed cases did opposite signs result. In the Table, only significant parts of the calculated figures are shown. The quantity  $q_1$  is the distance of nearest approach of the passing star to the sun and  $e_1$  is the eccentricity of the hyperbolic orbit of the star.

Table 1

Orbital elements of the planet after an encounter

$$q_1 = 0.5$$

$$e_1 = 20$$

| $\psi$                                                                      | a      | e      | i |
|-----------------------------------------------------------------------------|--------|--------|---|
| Initial orbital plane of the planet coplanar with that of the passing star. |        |        |   |
| 30°                                                                         | 0.9091 | 0.109  | 0 |
| 90°                                                                         | 1.315  | 0.330  | 0 |
| 150°                                                                        | 0.8475 | 0.3254 | 0 |
| 210°                                                                        | 2.024  | 0.5115 | 0 |
| 270°                                                                        | 0.7905 | 0.7785 | 0 |
| 328°                                                                        | 1.182  | 1.67   | 0 |
| 330°                                                                        | 1.036  | 1.46   | 0 |
| 332°                                                                        | 1.093  | 1.88   | 0 |

Initial orbital plane of the planet perpendicular to that of the passing star.

|      |        |        |             |
|------|--------|--------|-------------|
| 30°  | 5.974  | 0.8330 | 1° 14' 56"  |
| 90°  | 1.694  | 0.5302 | 1° 10' 30"  |
| 150° | 0.5675 | 0.4787 | -0° 57' 40" |
| 210° | 0.7407 | 0.4248 | -0° 32' 03" |
| 270° | 1.802  | 0.7004 | 1° 42' 49"  |
| 330° | 2.641  | 0.7353 | 0° 13' 31"  |

$$q_1 = 1$$

$$e_1 = 40$$

Initial orbital plane of the planet coplanar with that of the passing star.

|      |           |         |   |
|------|-----------|---------|---|
| 0°   | 0.7931    | 0.3251  | 0 |
| *90° | 0.2290    | 4.94    | 0 |
| 180° | 0.9402    | 0.07826 | 0 |
| 270° | Collision |         |   |

Initial orbital plane of the planet perpendicular to that of the passing star.

|      |        |        |             |
|------|--------|--------|-------------|
| 0°   | 1.094  | 0.2143 | -0° 19' 43" |
| 90°  | 1.181  | 0.1745 | 0° 36' 54"  |
| 180° | 0.8074 | 0.2410 | -0° 49' 30" |
| 270° | 40.8   | 1.024  | 2° 43' 21"  |

Table 1 (continued)

 $q_1 = 1.5$  $e_1 = 60$ 

| $\psi$                                                                      | $a$    | $e$     | $i$ |
|-----------------------------------------------------------------------------|--------|---------|-----|
| Initial orbital plane of the planet coplanar with that of the passing star. |        |         |     |
| $0^\circ$                                                                   | 1.037  | 0.04732 | 0   |
| $*90^\circ$                                                                 | 0.7758 | 0.2742  | 0   |
| $180^\circ$                                                                 | 1.139  | 0.1521  | 0   |
| $270^\circ$                                                                 | 0.9003 | 0.1906  | 0   |

Initial orbital plane of the planet perpendicular to that of the passing star.

|             |        |        |                     |
|-------------|--------|--------|---------------------|
| $90^\circ$  | 0.9064 | 0.1907 | $-0^\circ 30' 02''$ |
| $270^\circ$ | 0.8034 | 0.2570 | $-0^\circ 34' 50''$ |

 $q_1 = 2$  $e_1 = 80$ 

Initial orbital plane of the planet coplanar with that of the passing star.

|             |        |         |   |
|-------------|--------|---------|---|
| $0^\circ$   | 1.018  | 0.02605 | 0 |
| $*90^\circ$ | 0.8587 | 0.1487  | 0 |
| $180^\circ$ | 1.049  | 0.06412 | 0 |
| $270^\circ$ | 0.9567 | 0.07978 | 0 |

Initial orbital plane of the planet perpendicular to that of the passing star.

|             |        |         |                     |
|-------------|--------|---------|---------------------|
| $90^\circ$  | 0.9389 | 0.06928 | $-0^\circ 23' 27''$ |
| $270^\circ$ | 0.8728 | 0.1479  | $-0^\circ 0' 12''$  |

(iii) Estimate of numerical errors. It is rather difficult to estimate accurately the error that occurs owing to the fact that we did not extend the integrations from minus infinity to plus infinity. However since the initial conditions are specified without uncertainty, the error activity involved arises from the fact that the integration is not extended to plus infinity. For almost all cases the energy value  $E$  approaches a constant value when  $R_1$ , the distance between the sun and the passing star become 10. Since step-by-step integration is stopped at  $R_1 \sim 10$  for almost all cases, the errors can be estimated in the following way. The equations for the variation of the elements are, by (5.1), (6.1) and (7.1),

$$\begin{aligned} \frac{da}{dt} &= \frac{2\gamma M_1}{na} \left[ ae \sin u \frac{2a}{R_1^3} P_2(S) + \dots \right] \\ \frac{de}{dt} &= \frac{\gamma M_1}{na} \left[ \frac{(1-e^2)^{\frac{1}{2}}}{1-e \cos u} (1-e^2)^{\frac{1}{2}} \frac{2\gamma}{R_1^3} P_2(S) \sin u + \dots \right] \quad (10.3) \\ \frac{di}{dt} &= \frac{\gamma M_1}{na^2(1-e^2)} \left[ \frac{r}{R_1} P_2(S) + \dots \right] \sin i \end{aligned}$$

When the approximation  $dt = \frac{e_1}{n_1} \cosh D \cdot dD$ , and taking  $a/R_1 < 0.1$ ,  $e_1 = 20$ , we find that  $\delta a/a$ ,  $\delta e$ , and  $\delta i$  are of order  $10^{-4}$ . Since the eccentricity and the inclination of the planetary orbit are themselves of order  $10^{-1}$  and  $10^{-2}$ , respectively (as are seen from the examples in the Table), these elements clearly involve rather large relative errors as compared with the error of the semi-major axis.

The error due to the Runge-Kutta-Gill method is of the order the fifth power of the step-length. The adopted step-length is 0.01, and the

numerical error at each step of integration is of order  $10^{-10}$ , while the accuracy of the machine is of order  $10^{-8}$  or  $10^{-9}$ . Since a complete integration consists of about 800 steps, the cumulative error is of order  $10^{-5}$ . These errors are therefore less than the error arising from the fact that the integration was not carried out to plus infinity.

11. Cosmogonical implications. - The problem whether capture, and the reverse process of expulsion is possible in three-body motion has been considered by many authors, and in recent years it has become decisively established that it is possible. Here expulsion is defined as a process in which two point masses are in relative elliptic motion and these masses are disturbed by a third body initially having hyperbolic velocity at infinity with respect to the binary system; the binary system is disrupted by the third body, and after the encounter the three bodies separate to infinity in divergent directions. Since the equations of motion are completely reversible in time, expulsion is equivalent to the capture process.

At one time opinion was expressed that such capture would not occur, but Schmidt (8) found a numerical example of the process and thereby demonstrated its possibility. His example involved three finite masses, while a second example found by Sizova (9) is concerned with two finite masses and one infinitely small mass. The orbital elements in Sizova's example are, in our notation,

$a = 30$  a.u.,  $e = 0$ , for the massless particle relative to the sun,

$q_1 = 30$  a.u.,  $e_1 = 38.68$ , for the hyperbolic disturbing star.

Thus Sizova's example is quite similar to the second of the present examples (Table p. 29) for  $q_1 = 1$ .

On comparing the numerical examples with the analytical solutions for  $q_1 = 10 \underline{a}$ , we find that the disturbance due to a close encounter can be very much greater, as would be expected. Thus our conclusion about the stability of the solar system is that although the system is stable against the cumulative effect of distant stellar encounters, it is not always stable against a close encounter. However it is to be noted that, as far as the numerical examples carried out are concerned, expulsion of a planet does not occur owing to disturbing passing star if the distance of closest approach is greater than the semi-major axis  $\underline{a}$  of the planetary orbit before the encounter. The numerical examples show that a star can pass near the solar system and only disturb the outer most planet, without causing any significant changes in the orbits of the inner-planets, if initial conditions are properly satisfied.

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